Estimation of the Effect of Multiplicative Noise on Signal Resolution Conditions Using a Statistical Criterion

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***Abstract*—We analyzed the effect of multiplicative noise on signal resolution characteristics when a statistical criterion for processing is used in the receiving device. The receiver is optimal for resolving two signals in the sense of detecting against the background of additive white noise. It is shown that the complex envelope of the signal at the output of the analyzed receiver can be considered as a flat vector with correlated components having different variances. However, with fast multiplicative noise, the named components become uncorrelated, and their variances become equal. It is shown that the delay time resolution intervals and frequency shift resolution intervals with known statistical characteristics of signals, additive and multiplicative noise are uniquely determined by the probabilities of correct and false resolution. It is pointed out that the use of a statistical criterion allows us to determine quantitative values of resolution intervals in a fairly wide range of changes in the parameters of additive and multiplicative noise, significantly exceeding the Woodward criterion for example. Thus, the use of statistical criteria leads to an increase in the efficiency of radio systems.**

Keywords*—receiver, fast multiplicative noise, additive white noise, statistical criterion, resolution interval, resolution probability, Woodward criterion

I. Introduction

We estimate the effect of multiplicative (modulating) noise (MN) on the signal resolution characteristics during their processing in the receiver, optimal for the resolution of two signals in terms of detection, against the background of additive noise (AN) in the form of white noise [1, 2, etc.].

We assume that the input of the receiver can receive two signals with random initial phases and unknown amplitudes, which differ in the delay time by τ and in the frequency shift by Ω. In this case, it is known a priori that one of the signals, which will further be called the first one, with a complex envelope $\dot{U}_1(t)$ is present, and its parameters τ and Ω are equal to τ_0 and Ω_0, respectively. Without violating generality, it is assumed that $\tau_0 = 0,\ \Omega_0 = 0$. The second signal with a complex envelope $\dot{U}_2(t-\tau,-\Omega)$ differs from the first one in the amplitude, time shift τ, and frequency shift Ω that are considered to be known. Additive white noise with a complex envelope $\dot{N}(t)$ can be found at the input of the receiver simultaneously with the two signals.

If the probability of detecting signal 2 is almost independent of the presence of signal 1, then it is considered that it is resolved relative to the first signal in the sense of detection [2, 4]. The probability of detecting signal 2 in this case can serve as a quantitative measure of the resolution of two signals, depending on the parameters τ and Ω, the level of signals and noise. They will be called the probability of resolution P_r.

The receiving device, which is optimal for the resolution, in terms of detection, of signal 2 against the background of signal 1 and AN as white noise, forms an output effect in the form of [2, 5]

$$Z = \frac{1}{2}\left|\int_{-\infty}^{\infty} \dot{U}_{in}(t) Q^*(t)\,dt\right|, \qquad (1)$$

where

$$\dot{U}_{in}(t) = \dot{U}_1(t) + \dot{U}_2(t-\tau,-\Omega) + \dot{N}(t) =$$
$$= \sqrt{E_1}\dot{U}_0(t) + \sqrt{E_2}\dot{U}_0(t-\tau)\exp\{-j\Omega(t-\tau)\} + \dot{N}(t)$$

is a complex envelope of the input mixture of two signals and normal white noise $\dot{N}(t)$; $\dot{Q}(t)$ is a complex envelope of the reference signal of the correlation receiver; $\dot{U}_0(t)$ is a complex envelope of the signal that differs from the received one in that its energy is equal to unity; E is signal energy.

Note that here and further an asterisk sign will mean a complex conjugate.

As mentioned above, the probability of resolution of signal 2 relative to signal 1 in the adopted model is determined by the probability of its detection against the background of signal 1 and AN. It means that it is determined by the probability of the output effect Z exceeding a certain threshold level Z_{th}, which is selected based on the given probability of false detection of the signal $\dot{U}_2$ in its absence. This probability is formally defined in the same way as false alarm probability in the detection theory. Let us denote it as F_r. Further, it will be called the false alarm probability.

Multiplicative noise distorts both received signals. In general, the noise modulation functions (NMF) of the first and second signals may be different. We denote these functions $\dot{M}_1(t),\ \dot{M}_2(t)$ respectively, and assume them to be stationary and stationary connected by random functions.

Note that $\dot{M}(t) = \eta(t)\exp\{i\varphi(t)\}$ is the NMF, which fully characterizes the parasitic modulation of the signal; $\eta(t) = \eta_0[1+\xi(t)] \geq 0$ is a dimensionless multiplier, which characterizes changes in the signal envelope caused by MN (amplitude distortion); η_0 is mathematical expectation $\eta(t)$; $\xi(t)$ is a stationary random process with zero mean $[1+\xi(t)] \geq 0$; $\varphi(t)$ are changes in the signal phase caused by MN (phase distortion).

With the above in mind, when MN is present, the complex envelope of the signal at the input of the receiver in question will be written as

$$\dot{U}_{in.m}(t) = \dot{M}_1(t)\sqrt{E_1}\dot{U}_0(t) + \dot{M}_2(t)\sqrt{E_2}\dot{U}_0(t-\tau)\times \times\exp\{-j\Omega(t-\tau)\} + \dot{N}(t). \quad (2)$$

Further, we will consider the case when MN has a noticeable effect on the probability of resolution, while in the absence of MN, signals 1 and 2 are confidently resolved. It is obvious that the autocorrelation function of the signal $\dot{\rho}(\tau,\Omega)$ will be much less than unity $|\dot{\rho}(\tau,\Omega)| << 1$.

II. Estimation of the Influence of Multiplicative Noise on Signal Resolution Conditions Based on a Statistical Criterion

As it was stated before, it is necessary to introduce a Taking into account the defined conditions, the reference signal $\dot{Q}(t)$ for estimating the influence of MN can be approximately written as

$$\dot{Q}(t) \simeq \dot{U}_0(t-\tau)\exp\{-j\Omega(t-\tau)\}. \quad (3)$$

Substituting (2) in (1) and taking into account (3), we obtain an expression for a complex envelope of the output effect of the linear part of the receiver in the presence of additive and multiplicative noise:

$$\dot{Z}_m = \frac{1}{2}\sqrt{E_2}\int_{-\infty}^{\infty}\dot{M}_2(t)\left|\dot{U}_0(t-\tau)\right|^2 dt + \exp\{-j\Omega\tau\}\frac{1}{2}\sqrt{E_1}\times \times\int_{-\infty}^{\infty}\dot{M}_1(t)U_0(t)U_0^*(t-\tau)\exp\{j\Omega t\}dt + \exp\{-j\Omega\tau\}\times \times\frac{1}{2}\int_{-\infty}^{\infty}\dot{N}(t)U_0^*(t-\tau)\exp\{j\Omega t\}dt. \quad (4)$$

In accordance with (1), the output effect of the receiver in the presence of MN is determined by the ratio

$$Z_m = |\dot{Z}_m| = \sqrt{\left[\operatorname{Re}\{\dot{Z}_m\}\right]^2 + \left[\operatorname{Im}\{\dot{Z}_m\}\right]^2} = \sqrt{x^2+y^2}.$$

In order to find the resolution probability F_r, first of all, it is necessary to determine the statistical characteristics of the quadrature components x and y of the complex envelope of the signal at the output of the linear part of the receiver.

Given that $\overline{\dot{M}_1(t)} = \alpha_{01}\exp\{j\beta_{01}\}$, $\overline{\dot{M}_2(t)} = \alpha_{02}\exp\{j\beta_{02}\}$, where, α_0, β_0 are, respectively, a relative level of the undistorted part of the signal and its initial phase, and assuming that $\beta_{01} = \beta_{02} = 0$, we obtain the following expression for the average value $\dot{Z}_m$:

$$\overline{\dot{Z}_m} = \overline{\operatorname{Re}\dot{Z}_m} + j\,\overline{\operatorname{Im}\dot{Z}_m} = = \alpha_{02}\sqrt{E_2} + \alpha_{01}\sqrt{E_1}\exp\{-j\Omega\tau\}\dot{\rho}(\tau,\Omega). \quad (5)$$

Note that the variance of the quadrature components and their cross-correlation function can be written as

$$\left.\begin{aligned}\sigma_x^2 &= \frac{1}{2}\overline{\operatorname{Re}\dot{Z}_0'\,Z_0^{*\prime\prime}} + \frac{1}{2}\overline{\operatorname{Re}\dot{Z}_0'\,Z_0^{*\prime\prime}};\\ \sigma_y^2 &= \frac{1}{2}\overline{\operatorname{Re}\dot{Z}_0'\,Z_0^{*\prime\prime}} - \frac{1}{2}\overline{\operatorname{Re}\dot{Z}_0'\,Z_0^{*\prime\prime}},\end{aligned}\right\} \quad (6)$$

where $\dot{Z}_0' = \dot{Z}_m' - \overline{\dot{Z}_m'}$; $\dot{Z}_0'' = \dot{Z}_m'' - \overline{\dot{Z}_m''}$.

Superscripts (character strokes) for $\dot{Z}_m$ and $\dot{Z}_0$ indicate the difference in the integration variables in (4). That is, Z_m' is defined by the expression (4) for the integration variable t', while Z_m'' is defined for the variable t''.

If we express $\dot{Z}_0' = \dot{b}_1' + \dot{b}_2' + \dot{b}_3'$, where $\dot{b}_i'$ are corresponding centered terms of the right-hand side of (4), then

$$\left.\begin{aligned}\dot{Z}_0'\,Z_0^{*\prime\prime} &= \sum_{i=1}^{3}\sum_{k=1}^{3}\dot{b}_i'\,b_k^{*\prime\prime};\\ \dot{Z}_0'\,Z_0'' &= \sum_{i=1}^{3}\sum_{k=1}^{3}\dot{b}_i'\,b_k''.\end{aligned}\right\} \quad (7)$$

After calculating the formula (7), we get

$$\overline{\dot{Z}_0'\,Z_0^{*\prime\prime}} = 2E_2\delta_{12}^2(0,0) + 2E_1\delta_{11}^2(\tau,\Omega) + +\sqrt{E_1E_2}\operatorname{Re}\{\exp\{j\Omega\tau\}B_{s.21}(0,\tau,0,\Omega)\} + N_0; \quad (8)$$

$$\dot{Z}_0'\,\dot{Z}_0'' = 2E_2\delta_{22}^2(0,0) + 2E_1\delta_{21}^2(\tau,\Omega) + +\sqrt{E_1E_2}\operatorname{Re}\{\exp\{-j\Omega\tau\}D_{s.21}(0,\tau,0,\Omega)\}. \quad (9)$$

The following notation is introduced here:

$$\left.\begin{aligned}&\delta_{1i}^2(\tau,\Omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty}G_{Vi}(\omega)\left|\dot{\rho}(\tau,\Omega+\omega)\right|^2 d\omega;\\ &\delta_{2i}^2(\tau,\Omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\dot{G}_{Di}(\omega)\dot{\rho}(\tau,\Omega+\omega)\dot{\rho}(\tau,\Omega-\omega)d\omega;\\ &B_{s.21}(0,\tau,0,\Omega) = \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}G_{V21}(\omega)\dot{\rho}(0,\omega)\rho^*(\tau,\Omega+\omega)d\omega;\\ &D_{s.21}(0,\tau,0,\Omega) = \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\dot{G}_{D21}(\omega)\dot{\rho}(0,\omega)\dot{\rho}(\tau,\Omega-\omega)d\omega,\end{aligned}\right\} \quad (10)$$

where $G_{V1}(\omega)$ is the energy spectrum of fluctuations $\dot{M}_1(t)$; $G_{V21}(\omega)$ is the mutual energy spectrum of differences $\dot{M}_1(t) - \alpha_{01}\exp\{j\beta_{01}\}$ and $\dot{M}_2(t) - \alpha_{02}\exp\{j\beta_{02}\}$; $\dot{G}_{D21}$ is the Fourier transform of the function

$$\dot{G}_{V21}(t_1-t_2) = = \overline{\left[\dot{M}_2(t_1) - \alpha_{02}\exp\{j\beta_{02}\}\right]\left[\dot{M}_1(t_2) - \alpha_{01}\exp\{j\beta_{01}\}\right]}.$$

Expressions for $\delta_{12}^2(\tau,\Omega)$, $\delta_{22}^2(\tau,\Omega)$, that characterize signal 2 have the form similar to $\delta_{11}^2(\tau,\Omega)$, $\delta_{21}^2(\tau,\Omega)$.

Analyzing expressions (6)–(10), we see that the complex envelope of the signal at the output of the analyzed receiver can be considered as a flat vector with correlated components having different variances. However, in cases where MN leads to a significant deterioration of the resolution conditions, that is, with relatively fast and strong MN, the quadrature components x and y become uncorrelated random processes close to normal, with equal variances.

Similar results are also obtained in cases where the NMF $\dot{M}_1(t)$, $\dot{M}_2(t)$ are stationary normal random processes. In both of these cases, the probability density of the output effect of the receiver Z_m can be considered as generalized Rayleigh probability density:

$$W(Z_m)=\frac{Z_m}{\sigma^2}\exp\left\{-\frac{Z_m^2+m^2}{2\sigma^2}\right\}I_0\left(\frac{mZ_m}{\sigma^2}\right)$$

with the parameters: $m=\mathrm{Re}\{\overline{\dot{Z}_m}\}$, where $\overline{\dot{Z}_m}$ is determined from the expression (5), and

$$\sigma^2=\sigma_x^2=\sigma_y^2=E_2\delta_{12}^2(0,0)+E_1\delta_{11}^2(\tau,\Omega)+ +\frac{1}{2}\sqrt{E_1E_2}\,\mathrm{Re}\{\exp\{j\Omega\tau\}B_{s.21}(0,\tau,0,\Omega)\}+N_0/2. \quad (11)$$

It should be noted, that for relatively fast MN, when the quadrature components x and y can be assumed to be normal, the coefficient of mutual correlation of signals $B_{s.21}(0,\tau,0,\Omega)$ at such values τ, Ω, at which the probability of resolution in the absence of MN is high enough , is small. Thus, the third term in (11) in comparison with the first two terms can be neglected. Finally, we get

$$\sigma^2=E_2\delta_{12}^2(0,0)+E_1\delta_{11}^2(\tau,\Omega)+N_0/2. \quad (12)$$

The probability of resolution of the second signal relative to the first one in the presence of additive and multiplicative noise, taking into account (11), will be

$$P_r=\int_{Z_\text{п}}^{\infty}\frac{Z_m}{\sigma^2}\exp\left\{-\frac{Z_m^2+m^2}{2\sigma^2}\right\}I_0\left(\frac{mZ_m}{\sigma^2}\right)dZ_m= =Q\left(\frac{m}{\sigma},\frac{Z_{th}}{\sigma}\right), \quad (13)$$

where $Q\left(\frac{m}{\sigma},\frac{Z_{th}}{\sigma}\right)$ is a Q-function; Z_{th} is a the threshold value Z, which is determined based on the specified level F_r of probability of false alarm (false detection) of signal 2 against the background of signal 1 and noise.

When determining the probability of false alarm, one can take the distribution of the output effect Z as simple Rayleigh distribution ($m = 0$), then the threshold is

$$Z_{th}=\sigma_1\sqrt{2\ln F_r^{-1}}, \quad (14)$$

where $\sigma_1^2=E_1\delta_{11}^2(\tau,\Omega)+N_0/2$.

The relations (13) and (14) can be used to estimate the effect of MN on the resolution of two signals. They show that the probability of resolution of signal 2 is defined as the probability of detecting the sum of undistorted part and the noise components of the second signal exposed to AN and the noise component of the first signal at the point, where the undistorted part of the second signal is situated.

There are cases where the ratio of the power of the undistorted part of the signal at the output of the linear part of the receiver m^2 to the power of fluctuations caused by AN and MN (12), while σ^2 at the point where signal 2 is located is greater than unity. In these cases, the resolution probability can be expressed in terms of the well-known and properly tabulated Laplace function (probability integral).

Taking into account [5] the asymptotic formula, the expression (13) is transformed into

$$P_r=1-F(\gamma), \quad (15)$$

where

$$\left.\begin{aligned}\gamma&=u-\frac{1}{\frac{2m}{\sigma}}+\frac{u}{4\left(\frac{m}{\sigma}\right)^2}-\frac{u^2+0,5}{6\left(\frac{m}{\sigma}\right)^3};\\ u&=\frac{Z_{th}}{\sigma}-\frac{m}{\sigma}=\frac{\sigma_1}{\sigma}\sqrt{2\ln F_r^{-1}}-\frac{m}{\sigma};\end{aligned}\right\} \quad (16)$$

$$F(\gamma)=\frac{1}{\sqrt{2\pi}}\int_\gamma^{-\infty}\exp\left\{-\frac{x^2}{2}\right\}dx. \quad (17)$$

From subsequent expressions, it follows that the approximate formula (15), as an asymptotic representation, is valid only when $m/\sigma>1$, that is, in cases when the peak power of the undistorted part of the signal exceeds the power of fluctuations due to MN and AN at the point where the signal 2 is located.

If the probabilities of correct resolution and false alarm P_r, F_r are given, then based on (15)–(17), the integral of delay resolution τ_r or frequency resolution Ω_r corresponding to the given probabilities P_r, F_r can be determined.

Apparently, the power of fluctuations σ^2, σ_1^2 ((12), (14)) depends on the shift between signals in time and frequency, moreover

$$\sigma^2(\tau,\Omega)=\sigma_1^2(\tau,\Omega)+E_2\delta_{12}^2(0,0).$$

Thus, to find the resolution intervals for the given values P_r, F_r it is necessary to find the function $\sigma_1^2(\tau,\Omega)=E_1\delta_{11}^2(\tau,\Omega)+N_0/2$ from (15)–(17) taking into account (14), and then, having calculated $\delta_{11}^2(\tau,\Omega)$, determine the resolution intervals.

Let us denote the argument value γ by $\gamma_0(P_r)$. It satisfies equation (15) for the given value P_r. The equation for finding σ_1^2, which is obtained by substituting $\gamma_0(P_r)$ for γ in the left part of (16), is very complex and its solution can be found only by numerical methods. We will only calculate the first approximation for $\sigma_1^2(\tau,\Omega)$, which is valid only for large relations m/σ. Keeping the first two terms of the function $\gamma(u)$ on the right side of (16), we have

$$\gamma_0=\frac{Z_{th}}{\sigma}-\frac{m}{\sigma}\left(1+\frac{1}{2m^2/\sigma^2}\right). \quad (18)$$

If it is assumed $2m^2/\sigma^2>>1$ and the notation $\sigma^2=\sigma_1^2+a^2$; $Z_{th}=\sigma_1 b$ is introduced, we get the following quadratic equation with respect to σ_1:

$$\sigma_1^2\left(\gamma_0^2-b^2\right)+2\sigma_1 bm+\gamma_0^2 a^2-m^2=0, \tag{19}$$

where $a^2=E_2\delta_{12}^2(0,0)$; $b^2=2\ln\left(1/F_r\right)$.

Note, that the parameters a^2 and b are determined by the statistical characteristics of the MN, the specified false alarm level, and do not depend on the intervals τ, Ω between signals.

The equation (19) is solved as

$$\sigma_1^2=\left[\frac{bm-\sqrt{b^2m^2+\left(\gamma_0^2a^2-m^2\right)\left(b^2-\gamma_0^2\right)}}{b^2-\gamma_0^2}\right]^2. \tag{20}$$

By substituting the calculated value σ_1^2 in (14), the resolution intervals τ_r, Ω_r can be determined from a known function $\delta_{11}^2(\tau,\Omega)$. In many cases, there are simple functions inverse to $\delta_{11}^2(\tau,\Omega)$. Denoting the function inverse to $\delta_{11}^2(\tau,\Omega)$ as $\operatorname{arc}\delta_{11}^2(\tau,\Omega)$, we get

$$\tau_r,\Omega_r=\operatorname{arc}\delta_{11}^2\left(\frac{\sigma_1^2}{E_1}-\frac{N_0}{2E_1}\right). \tag{21}$$

For example, if $\delta_{11}^2(\tau,0)=\exp\left\{-\alpha^2\tau^2\right\}$, then $\operatorname{arc}\delta_{11}^2=\frac{1}{a}\sqrt{-\ln\delta_{11}^2}$, so the resolution intervals are written as

$$\tau_r,\Omega_r=\frac{1}{a}\sqrt{-\ln\left(\frac{\sigma_1^2}{E_1}-\frac{N_0}{2E_1}\right)}.$$

Let us consider another method for calculating resolution intervals based on the simplest linear approximation of the relation between the squares of the arguments of an incomplete Toronto function [6], with the Q-function being its special case

$$Q\left(x\sqrt{2},y\sqrt{2}\right)=1-T_y(1,0,x),$$

where $T_y(1,0,x)$ is the incomplete Toronto function. Note that in our problem $x^2=m^2/2\sigma^2$, $y^2=Z_{th}^2/2\sigma^2$.

Figure 1 shows the dependencies $y^2=f\left(x^2\right)$ for several fixed values P_r (solid line) and linear approximation (dotted line) of these dependencies by the functions

$$y^2=\mu x^2-\nu. \tag{22}$$

It is illustrated by figure 1, that the functions of type (22) effectively approximate real dependences at significant intervals of change in values x^2 and y^2.

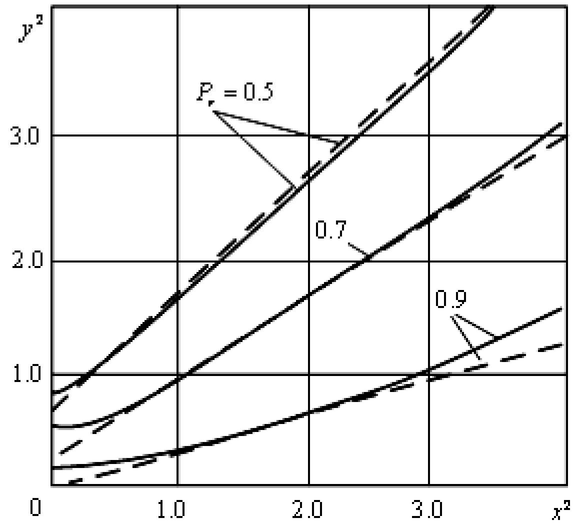

Fig. 1. Dependency $y^2=f\left(x^2\right)$

Table 1 shows the values of the coefficients μ and ν for several probabilities of correct resolution P_r and the boundary values x and y for which the introduced approximation is valid.

For each P_r value, there are two values of the coefficients μ and ν, and, consequently, there are two values of the interval bounds for approximation applicability (22).

The dependencies shown in figure 1 correspond to the values μ and ν specified for each P_r in the top row. When changing x and y within the intervals specified in table 1, the relative errors in fulfilling the conditions $P_r=\text{const}$ based on which the dependencies $y^2=f\left(x^2\right)$ are constructed, are several percent.

Taking into account (22), we can write the following relation for finding resolution intervals:

$$\frac{\sigma_1^2b^2}{\sigma_1^2+a^2}=\mu\frac{m^2}{\sigma_1^2+a^2}-2\nu, \tag{23}$$

from where, using (14), (21), as well as expressions for parameters a^2 and b from (19), we get

$$\tau_r,\Omega_r\simeq\operatorname{arc}\delta_{11}^2\left[\frac{\mu\frac{m^2}{E_1}-2\nu\frac{E_2}{E_1}\delta_{12}^2(0,0)}{2\ln\left(\frac{1}{F_p}\right)+2\nu}-\frac{1}{q_1^2}\right], \tag{24}$$

where $q_1^2=2E_1/N_0$.

TABLE I. COEFFICIENT VALUES

P_r	μ	ν	x_{min}	x_{max}	y_{min}	y_{max}
0,5	1,25	0	1,41	3,74	1,58	3,74
	1	-0,7	0	2,45	0,89	2,55
0,7	0,925	0,6	1,58	3,6	1,41	3,4
	0,69	-0,2	0,7	2	0,7	1,7
0,9	0,75	1,5	2	4,5	1,2	3,6
	0,33	0	0,7	1,73	0,39	1,0
0,99	0,55	5	3,6	5	1,41	3

Expression (23) is much simpler than (18), but it can only be used with restrictions on the range of parameters x, y specified above, as well as in table 1. Since the parameters x and y are defined through the characteristics of the AN and MN and the characteristics of the signals, the relations below set implicit restrictions on the characteristics of signals and noise, under which the expression (24) is valid:

$$x_{\min}^2 \le \frac{m^2}{2\sigma^2} = \frac{m^2/N_0}{q_2^2\delta_{12}^2(0,0)+q_1^2\delta_{11}^2(\tau_r,\Omega_r)} \le x_{\max}^2, \quad (25)$$

$$y_{\min}^2 \le \frac{Z_{th}^2}{2\sigma^2} =$$

$$= 2\ln\left(\frac{1}{F_r}\right)\frac{q_1^2\delta_{11}^2(\tau_r,\Omega_r)+1}{q_2^2\delta_{12}^2(0,0)+q_1^2\delta_{11}^2(\tau_r,\Omega_r)+1} \le x_{\max}^2,$$

where $q_2^2 = 2E_2/N_0$.

It follows from (20), (24) that the resolution intervals τ_r, Ω_r with known statistical characteristics of signals, additive noise and modulating noise, are uniquely determined by the probabilities of correct and false resolution. These probabilities are taken into account in the specified formulas by the parameters γ_0, b in (20) and the parameters μ and ν in (24).

Approximate relations (20), (24) allow us to determine the quantitative values of resolution intervals in a quite wide range of changes in the parameters of additive and modulating noise. At the same time, the simplest criterion for quantifying the resolution intervals of two signals of the same intensity in the presence of MN is based on the Rayleigh concept of resolution (Woodward's criterion [6]), and can be used only under rather strict restrictions imposed on noise. In particular, it does not allow us to take into account the influence of AN on the resolution conditions, whereas with MN it gives correct results only in cases when the power of the undistorted part of the signal is small compared to the power of its noise component at the output of the linear part of the receiver [7].

III. Quantitative Restrictions on the Level of Additive Noise and Characteristics of Multiplicative Noise

It is of interest to quantify the restrictions on the level of AN and the characteristics of MN, in which the resolution intervals calculated on the basis of the Woodward criterion and on the basis of the simplest statistical criterion discussed above are quite close. Since in our analysis we will be interested in the case of a small level of the undistorted part of the signal, only the ratio (24) can be used in calculations to determine the resolution intervals.

We will determine the effect of the parameters α_0, q^2 on the resolution intervals for fixed values P_r, F_r. It is assumed that the resolution intervals determined based on the Woodward criterion and the statistical criterion coincide, when $\alpha_0^2 = 0$, $q^2 = \infty$, and are equal to τ_r, Ω_r. When $\alpha_0^2 = 0$, $q^2 = \infty$, the probability of resolution by the statistical criterion instead of (13) is determined by the ratio (26).

If the method of reference τ_r, Ω_r is chosen, based on the Woodward criterion, for example, it might be the width of a rectangle of the equivalent area, then when comparing the results obtained using the above-mentioned criteria, it is necessary to take into account that in accordance with (26), each probability value P_r, corresponds to a well-defined and unique value F_r (27).

$$P_r = \exp\left\{-\frac{Z_{th}^2}{2\sigma^2}\right\}\exp\left\{-\frac{\ln\left(\frac{1}{F_r}\right)}{1+\frac{\delta_1^2(0,0)}{\delta_1^2(\tau,\Omega)}}\right\}. \quad (26)$$

$$F_r = \exp\left\{-\left[1+\frac{\delta_1^2(0,0)}{\delta_1^2(\tau,\Omega)}\right]\ln\left(\frac{1}{P_r}\right)\right\}. \quad (27)$$

If the values P_r, F_r are set, then in order to determine the scope of the Woodward criterion in the presence of MN, it is necessary to choose a method for determining the values τ_r, Ω_r. based on this criterion. In other words, we have to determine such a reference level for the width of the interval occupied by the noise component at the output of the receiving device, in which the resolution intervals calculated using the Woodward criterion would coincide with the resolution intervals determined by the following formula, resulting from (26):

$$\tau_r,\Omega_r = \operatorname{arc}\delta_1^2\left[\delta_1^2(0,0)\frac{\ln P_r}{\ln\left(\frac{F_r}{P_r}\right)}\right].$$

В дальнейшем будем считать, что заданы вероятности правильного разрешения P_r и уровень отсчета, определяющий вудвордовские интервалы разрешения $\tau_{r.W}$, $\Omega_{r.W}$.

We will estimate the effect of the level of the undistorted part of the signal α_0^2 on the resolution intervals when the AN level is low and $2\xi/q^2(1-\alpha_0^2)\to 0$. For small values of Δ

$$\Delta \simeq \frac{\xi^2}{2\pi\lambda^2 l_{r.W}^2}\ln\left(\frac{\mu\frac{\alpha_0^2\xi}{1-\alpha_0^2}-\nu}{\ln(1/P_r)}\right). \quad (28)$$

Expression (28) is not directly included in the reference level at which the resolution interval must be measured when using the Woodward criterion. For the example under consideration, which relates the resolution interval $l_{r.W}$ to the reference level $\zeta = \delta_1^2(l_{р.в})/\delta_1^2(0)$:

$$\frac{2\pi\lambda^2 l_{r.W}^2}{\xi^2} = \ln\frac{\delta_1^2(0)}{\delta_1^2(l_{r.W})} = \ln\frac{1}{\zeta}. \quad (29)$$

Taking into account (29) instead of (28), we have

$$\Delta \simeq \frac{1}{\ln(1/\xi^2)}\ln\left(\frac{\mu\frac{\alpha_0^2\xi}{1-\alpha_0^2}-\nu}{\ln(1/P_r)}\right).$$

When the resolution interval is defined as the width of a rectangle equal to the area of the function $\delta_1^2(l)$, for the example under consideration, we have

$$\ln\frac{\delta_1^2(0)}{\delta_1^2(l_{r.W})}=\ln\frac{1}{\zeta}=\pi.$$

Inequality (28) shows that the relative error Δ in determining the resolution interval according to the Woodward criterion, which occurs due to the presence of the undistorted part of the signal, depends on the parameter $\alpha_0^2\xi/(1-\alpha_0^2)$. This parameter determines the ratio of the power of the undistorted part of the signal to the power of fluctuations caused by the presence of MN at the point where the undistorted part of the signal reaches its maximum.

Expression (28) also shows that the specified ratio, all other things being equal, can be the greater, the lower the reference level ζ determining the resolution interval $l_{r.W}$.

Now we will estimate the effect of AN on the resolution intervals, assuming $\alpha_0^2=0$. In order to do this, you must use a relation of (26) type as the original expression, since the expression (35) is not applicable if $\alpha_0^2=0$, since the condition (25) is not met.

From (26) taking into account (11), (14) we will get

$$l_r=\frac{\xi}{\lambda\sqrt{\pi}}\sqrt{-\ln\left(\exp\left\{\frac{\pi l_{r.W}\lambda^2}{\xi^2}-\frac{2\xi}{q^2}\right\}\right)}.$$

Expressing the value $l_{r.W}$ in terms of the reference level ζ, we finally get

$$l_r=\frac{\xi}{\lambda\sqrt{\pi}}\sqrt{-\ln\left(\exp\left\{\zeta-\frac{2\xi}{q^2}\right\}\right)}. \quad (30)$$

In (30), there is no explicit dependence of the resolution interval on the probability of correct resolution and the probability of false alarm. Obviously, this is due to the fact that for a given resolution interval $l_{r.W}$ based on the Woodward criterion, AN equally affects the resolution interval $l_{r.}$ for any values P_p, F_p that satisfy the relation (26).

The relative difference between the resolution intervals determined on the basis of the Woodward criterion and the statistical criterion (34), taking into account (30), will be

$$\Delta=1-\sqrt{1+\frac{1}{\ln\zeta}\ln\left(1-\frac{2\xi}{q^2\zeta}\right)}. \quad (31)$$

For small relative errors Δ, which we are interested in, we can expand the second term in (31) into a power series and restrict ourselves to two terms of the expansion. Then, representing the natural logarithm by two terms of its Taylor series expansion in the vicinity of the point $2\xi/q^2\zeta=1$, we will finally get

$$\Delta\simeq\frac{\xi}{q^2\zeta\ln\zeta}. \quad (32)$$

The formula (32) allows us to determine the acceptable level of AN at which it is still possible to use the Woodward criterion based on a given error value Δ.

It follows from the obtained formula, that the permissible ratio of the signal energy to the spectral density of power of additive noise q^2 is smaller, the narrower the spectrum of the noise modulation function and the smaller the parameter ξ.

If the resolution interval $l_{r.W}$ is defined as the equivalent width of the function describing the power distribution of signal fluctuations at the output of the linear part of the receiver, then $\zeta=\exp\{-\pi\}$, while the value $q^2\zeta/\xi$ must be at least not smaller than $(\pi|\Delta|)^{-1}$.

Since the parameter q^2 equal to the ratio of the peak power of the undistorted signal to the power of AN at the output of a coherent receiver, the value $q^2\zeta/\xi$ can be interpreted as the ratio of power of fluctuations of the signal distorted by MN at the point $l=l_{r.W}$ to the power of AN at the output of the same receiver. With this in mind, the use of the Woodward criterion for determining the difference intervals in the presence of AN does not lead to gross errors only in cases where the power of the noise component of the signal at the point $l=l_{r.W}$ exceeds the power of the AN.

IV. Conclusions

Relations are obtained that allow us to determine the quantitative values of the resolution intervals in a fairly wide range of changes in the parameters of additive and multiplicative noise. At the same time, the simplest criterion for quantifying the resolution intervals of two signals of the same intensity in the presence of multiplicative noise, based on the Rayleigh concept of resolution (Woodward's criterion), can be used only under sufficiently strict restrictions imposed on noise. Moreover, it does not allow us to take into account the influence of additive noise on the resolution conditions, whereas for multiplicative noise it gives correct results only in cases when the power of the undistorted part of the signal is small compared to the power of its noise component at the output of the linear part of the receiver. It is shown that the relative error in determining the resolution interval by the Woodward criterion, which occurs due to the presence of an undistorted part of the signal, depends on one particular parameter. This parameter determines the ratio of the power of the undistorted part of the signal to the power of fluctuations caused by the presence of multiplicative noise at the point where the undistorted part of the signal reaches its maximum.

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